Preference and Utility

Econ 2100 Fall 2025

Lecture 3

Outline

1. Existence of Utility Functions
2. Continuous Preferences
3. Debreu's Representation Theorem
4. Structural Properties of Utility Functions, Part I

Definitions From Last Week

- A binary relation $\succsim$ on X is a preference relation if it is a weak order, i.e. complete and transitive.
- The upper contour set of $\mathbf{x}$ is $\succsim(\mathbf{x}) = \{\mathbf{y} \in X : \mathbf{y} \succsim \mathbf{x}\}$.
- The lower contour set of $\mathbf{x}$ is $\precsim(\mathbf{x}) = \{\mathbf{y} \in X : \mathbf{x} \succsim \mathbf{y}\}$.
- The utility function $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X if

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow u(\mathbf{x}) \geq u(\mathbf{y}).$$

Question for Today.

- Under what assumptions can a preference relation be represented by a utility function?
    - We know transitivity and completeness are needed. Are they enough?

Existence of a Utility Function: Alternative Definition

- The following provides an alternative way to show that a preference is represented by some utility function.

Question 1, Problem Set 2.

Let $\succsim$ be a preference relation. Prove that $u : X \to \mathbb{R}$ represents $\succsim$ if and only if:

$$\mathbf{x} \succsim \mathbf{y} \Rightarrow u(\mathbf{x}) \geq u(\mathbf{y}); \qquad \text{and} \qquad \mathbf{x} \succ \mathbf{y} \Rightarrow u(\mathbf{x}) > u(\mathbf{y}).$$

- This result makes it (sometimes) easier to show a utility function represents $\succsim$.

Existence of a Utility Function When X is Finite

Theorem

Suppose X is finite. Then $\succsim$ is a preference relation if and only if there exists some utility function $u : X \to \mathbb{R}$ that represents $\succsim$.

- Finding a utility function that represents any given preference is easy when the space of outcomes is finite.
- The proof is constructive: a function that works is the one that counts the number of elements that are not as good as the one in question.
  - This function is well defined because the are only finitely many items that can be worse than something.
- In other words, the utility function is
$$u(\mathbf{x}) = |\precsim(\mathbf{x})|$$
where $\precsim(\mathbf{x}) = \{\mathbf{y} \in X : \mathbf{x} \succsim \mathbf{y}\}$ is the lower contour set of $\mathbf{x}$, and $|\cdot|$ denotes the cardinality of $\cdot$.
- Notice that we only need to prove one direction of the result since last class we established that if u is a utility function representing $\succsim$ then $\succsim$ is complete and transitive and thus is a preference relation.

Existence of a Utility Function When X is Finite

If X is finite and $\succsim$ is a preference relation $\Rightarrow \exists\ u : X \rightarrow \mathbb{R}$ that represents $\succsim$.

Proof.

Let $u(\mathbf{x}) = |\precsim(\mathbf{x})|$. Since X is finite, $u(\mathbf{x})$ is finite and therefore well defined.

- Suppose $\mathbf{x} \succsim \mathbf{y}$. I claim this implies $u(\mathbf{x}) \geq u(\mathbf{y})$.
  - Let $\mathbf{z} \in \precsim(\mathbf{y})$, i.e. $\mathbf{y} \succsim \mathbf{z}$.
  - By transitivity, $\mathbf{x} \succsim \mathbf{z}$, i.e. $\mathbf{z} \in \precsim(\mathbf{x})$. Thus $\precsim(\mathbf{y}) \subset \precsim(\mathbf{x})$.
  - Therefore $|\precsim(\mathbf{y})| \leq |\precsim(\mathbf{x})|$. By definition, this means $u(\mathbf{y}) \leq u(\mathbf{x})$.
- Now suppose $\mathbf{x} \succ \mathbf{y}$. I claim this implies $u(\mathbf{x}) > u(\mathbf{y})$.
  - $\mathbf{x} \succ \mathbf{y}$ implies $\mathbf{x} \succsim \mathbf{y}$, so the argument above implies $\precsim(\mathbf{y}) \subset \precsim(\mathbf{x})$.
  - $\mathbf{x} \succsim \mathbf{x}$ by completeness, so $\mathbf{x} \in \precsim(\mathbf{x})$. Also, $\mathbf{x} \succ \mathbf{y}$ implies $\mathbf{x} \notin \precsim(\mathbf{y})$.
  - Hence $\precsim(\mathbf{y})$ and $\{\mathbf{x}\}$ are disjoint, and both subsets of $\precsim(\mathbf{x})$. Then $\precsim(\mathbf{y}) \cup \{\mathbf{x}\} \subset \precsim(\mathbf{x})$

$$\begin{aligned} |\precsim(\mathbf{y}) \cup \{\mathbf{x}\}| &\leq |\precsim(\mathbf{x})| \\ |\precsim(\mathbf{y})| + |\{\mathbf{x}\}| &\leq |\precsim(\mathbf{x})| \\ u(\mathbf{y}) + 1 &\leq u(\mathbf{x}) \\ u(\mathbf{y}) &< u(\mathbf{x}) \end{aligned}$$

- This proves $\begin{array}{l} \mathbf{x} \succsim \mathbf{y} \Rightarrow u(\mathbf{x}) \geq u(\mathbf{y}) \\ \mathbf{x} \succ \mathbf{y} \Rightarrow u(\mathbf{x}) > u(\mathbf{y}) \end{array}$ and thus we are done. $\square$

Utility Function for a Countable Space

- Existence of a utility function can also be proven for a countable space.

Theorem

Suppose X is countable. Then $\succsim$ is a preference relation if and only if there exists some utility function $u : X \to \mathbb{R}$ that represents $\succsim$.

Proof.

Exercise □

- Notice that the previous construction cannot be applied directly here because the cardinality of the lower contour sets can be infinite.
- You will have to come up with a "trick" that works.

What Else?

- We want conditions on a preference relation which guarantee the existence of a utility function representing those preferences even if X is uncountable.
  - Typically, we assume that X lives in $\mathbb{R}^n$, but ideally we would want as few restrictions on X as we can get away with.
- We know that without completeness and transitivity a utility function does not exist, so we cannot dispense of those.
- What else is needed?

Lexicographic Preferences

Exercise

Show that the lexicographic ordering on $\mathbb{R}^2$ defined by

$$(x_1, x_2) \succsim (y_1, y_2) \quad \Leftrightarrow \quad \begin{cases} x_1 > y_1 \\ \text{or} \\ x_1 = y_1 \text{ and } x_2 \geq y_2 \end{cases}$$

is complete, transitive, and antisymmetric (i.e. if $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{x}$, then $\mathbf{x} = \mathbf{y}$).

- The lexicographic ordering is a preference relation (it is complete and transitive), yet admits no utility representation.

The Lexicographic Ordering in R^2 admits no utility representation

Let $X = \mathbb{R}^2$ and define $\succsim$ by $(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow \begin{cases} x_1 > y_1 \\ \text{or} \\ x_1 = y_1 \text{ and } x_2 \geq y_2 \end{cases}$

- Suppose there exists a utility function u representing $\succsim$.
- For any $x_1 \in \mathbb{R}$, $(x_1, 1) \succ (x_1, 0)$ and thus $u(x_1, 1) > u(x_1, 0)$ since u represents $\succsim$.
- The rational numbers are dense in the real line;[a] hence, there exists a rational number $r(x_1) \in \mathbb{R}$ such that: $$u(x_1, 1) > r(x_1) > u(x_1, 0) \quad \text{(A)}$$
- Define $r : \mathbb{R} \to \mathbb{R}$ by selecting $r(x_1)$ to satisfy A above.
- CLAIM: $r(\cdot)$ is a one-to-one function.
  - Suppose $x_1 \neq x_1'$; and without loss of generality assume $x_1 > x_1'$.
  - Then: $$r(x_1) \overbrace{>}^{by\ (A)} u(x_1, 0) \overbrace{>}^{\text{since } (x_1,0) \succ (x_1',1)} u(x_1', 1) \overbrace{>}^{by\ (A)} r(x_1')$$
- Thus $r(\cdot)$ is an a one-to-one function from the real numbers to the rational numbers, which contradicts the fact the real line is uncountable.

[a] For any open interval (a, b), there exists a rational number r such that $r \in (a, b)$.

Continuous Preferences

- Continuity will get rid of this example.

Definition

A binary relation $\succsim$ on the metric space X is continuous if, for all $\mathbf{x} \in X$, the upper and lower contour sets, $\{\mathbf{y} \in X : \mathbf{y} \succsim \mathbf{x}\}$ and $\{\mathbf{y} \in X : \mathbf{x} \succsim \mathbf{y}\}$, are closed.

Examples

- Define $\succsim$ by $(x_1, x_2) \succsim (y_1, y_2) \quad \Leftrightarrow \quad \begin{cases} x_1 > y_1 \\ \text{or} \\ x_1 = y_1 \text{ and } x_2 \geq y_2 \end{cases}$
  - Draw the upper contour set of $(1, 1)$; this preference on $\mathbb{R}^2$ is not continuous.
- Define $\succsim$ by $(x_1, x_2) \succsim (y_1, y_2) \quad \Leftrightarrow \quad \begin{cases} x_1 \geq y_1 \\ \text{and} \\ x_2 \geq y_2 \end{cases}$
  - Draw the upper contour set of $(1, 1)$; this preference on $\mathbb{R}^2$ is continuous.

Continuous Preferences

Definition

A binary relation $\succsim$ on the metric space X is continuous if, for all $\mathbf{x} \in X$, the upper and lower contour sets, $\{\mathbf{y} \in X : \mathbf{y} \succsim \mathbf{x}\}$ and $\{\mathbf{y} \in X : \mathbf{x} \succsim \mathbf{y}\}$, are closed.

Proposition

A binary relation $\succsim$ is continuous if and only if:

1. *If $\mathbf{x}_n \succsim \mathbf{y}$ for all n and $\mathbf{x}_n \to \mathbf{x}$, then $\mathbf{x} \succsim \mathbf{y}$; and*
2. *If $\mathbf{x} \succsim \mathbf{y}_n$ for all n and $\mathbf{y}_n \to \mathbf{y}$, then $\mathbf{x} \succsim \mathbf{y}$.*

Proof.

This follows from the fact that a set is closed if and only if it contains all of its limit points. □

Debreu's Representation Theorem

The main result of today is due to Gerard Debreu, and it provides necessary and sufficient conditions for the existence of a continuous utility function.

Theorem (Debreu)

Suppose $X \subseteq \mathbb{R}^n$. The binary relation $\succsim$ on X is complete, transitive, and continuous if and only if there exists a continuous utility representation $u : X \to \mathbb{R}$.

- We will prove sufficiency next (under a couple of simplifying assumptions).
- You will do necessity as homework.

Question 2, Problem Set 2.

Suppose $X \subseteq \mathbb{R}^n$. Prove that if $u : X \to \mathbb{R}$ is a continuous utility function representing $\succsim$, then $\succsim$ is a complete, transitive, and continuous preference relation.

Debreu's Representation Theorem

Theorem (Debreu)

Suppose $X \subseteq \mathbb{R}^n$. The binary relation $\succsim$ on X is complete, transitive, and continuous if and only if there exists a continuous utility representation $u : X \to \mathbb{R}$.

- Two simplifying assumptions for the proof:
  - $X = \mathbb{R}^n$; and
  - $\succsim$ is strictly monotone, (i.e. if $x_i \geq y_i$ for all i and $\mathbf{x} \neq \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$).
    - When strict monotonicity holds we have

$$\alpha \geq \beta \Leftrightarrow \alpha\mathbf{e} \succsim \beta\mathbf{e}. \quad \text{(mon)}$$

where $\mathbf{e} = (1, 1, \ldots, 1)$ (make sure you check this).

To prove:

if $\succsim$ is a continuous and strictly monotone preference relation on $\mathbb{R}^n$, then there exists a continuous utility representation of $\succsim$.

- How do we find a utility function? For each $\mathbf{x}$, look at the point on the 45^o line that is indifferent to it:

$$u(\mathbf{x}) = \alpha^*(\mathbf{x}) \text{ where } \alpha^*(\mathbf{x}) \text{ is the real number } \alpha^* \text{ such that } \alpha^*\mathbf{e} \sim \mathbf{x}$$

- The proof is in 3 steps.

Debreu's Representation Theorem Geometry

Step 1: There exists a unique $\alpha^*(\mathbf{x})$ such that $\alpha^*\mathbf{e} \sim \mathbf{x}$.

Proof.

Let $B = \{\beta \in \mathbb{R} : \beta\mathbf{e} \succsim \mathbf{x}\} \subset \mathbb{R}$ and define $\quad \alpha^* = \inf \underbrace{\{\beta \in \mathbb{R} : \beta\mathbf{e} \succsim \mathbf{x}\}}_{B}.$

- Oviously, $(\max_i x_i)\mathbf{e} \geq \mathbf{x}$, so by strict monotonicity $(\max_i x_i)\mathbf{e} \succsim \mathbf{x}$ which implies B is nonempty and α^* is well-defined.
- Next show that $\alpha^*\mathbf{e} \succsim \mathbf{x}$ and $\alpha^*\mathbf{e} \precsim \mathbf{x}$, so that $\alpha^*\mathbf{e} \sim \mathbf{x}$.
  - Since α^* is the infimum of B, there exists a sequence $\beta_n \in B$ s.t. $\beta_n \to \alpha^*$.
  - Then $\beta_n\mathbf{e} \to \alpha^*\mathbf{e}$ (in $\mathbb{R}^n$) and $\beta_n\mathbf{e} \succsim \mathbf{x}$ because $\beta \in B$.
  - By continuity, one gets: $\alpha^*\mathbf{e} \succsim \mathbf{x}$ as desired.
  - Since α^* is a lower bound of B, if $\alpha \in B \Rightarrow \alpha \geq \alpha^*$.
  - Using the contrapositive:
  $$\alpha < \alpha^* \Rightarrow \alpha\mathbf{e} \prec \mathbf{x}. \tag{A}$$
  - Let $\alpha_n = \alpha^* - \frac{1}{n}$. By (A), $\alpha_n\mathbf{e} \prec \mathbf{x}$, so $\alpha_n\mathbf{e} \precsim \mathbf{x}$.
  - Also, $\alpha_n\mathbf{e} \to \alpha^*\mathbf{e}$ (in $\mathbb{R}^n$). Hence, by continuity, $\alpha^*\mathbf{e} \precsim \mathbf{x}$ as desired.
- To prove uniqueness, suppose $\hat{\alpha}\mathbf{e} \sim \mathbf{x}$.
  - By transitivity, $\hat{\alpha}\mathbf{e} \sim \alpha^*\mathbf{e}$. Then, by (mon), $\hat{\alpha} = \alpha^*$. □

Step 2: Show that $u(\mathbf{x})$ defined as:

$$u(\mathbf{x}) = \alpha^*(\mathbf{x}) \qquad \text{where } \alpha^*(\mathbf{x}) \text{ is the unique } \alpha^* \text{ such that } \alpha^*\mathbf{e} \sim \mathbf{x}.$$

represents $\succsim$.

Proof.

We need to prove that $u(\mathbf{x})$ represents $\succsim$.

- Suppose $\mathbf{x} \succsim \mathbf{y}$.
  - By construction of $\alpha^*(\cdot)$, we have:
    $$\mathbf{x} \sim \alpha^*(\mathbf{x})\mathbf{e} \qquad \text{and} \qquad \alpha^*(\mathbf{y})\mathbf{e} \sim \mathbf{y}$$
  - By transitivity, we have:
    $$\mathbf{x} \sim \alpha^*(\mathbf{x})\mathbf{e} \succsim \alpha^*(\mathbf{y})\mathbf{e} \sim \mathbf{y}$$
  - By (mon) this is equivalent to
    $$\alpha^*(\mathbf{x}) \geq \alpha^*(\mathbf{y})$$
- Repeat the same argument to show that $\mathbf{x} \succ \mathbf{y}$ implies $\alpha^*(\mathbf{x}) > \alpha^*(\mathbf{y})$.
- Since we have shown that $\begin{array}{l} \mathbf{x} \succsim \mathbf{y} \Rightarrow u(\mathbf{x}) \geq u(\mathbf{y}) \\ \mathbf{x} \succ \mathbf{y} \Rightarrow u(\mathbf{x}) > u(\mathbf{y}) \end{array}$ we are done.

□

Step 3: Show that the defined $u(x)$ is continuous.

$$u(\mathbf{x}) = \alpha^*(\mathbf{x}) \quad \text{where } \alpha^*(\mathbf{x}) \text{ is the unique } \alpha^* \text{ such that } \alpha^*\mathbf{e} \sim \mathbf{x}$$

Proof.

Prove that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous by showing that $f^{-1}((a,b))$ is open for all $a, b \in \mathbb{R}$.

- Since $a\mathbf{e} \sim a\mathbf{e}$, we have $u(a\mathbf{e}) = \alpha^*(a\mathbf{e}) = a$ for any $a \in \mathbb{R}$.
- Notice that
$$u^{-1}((a,b)) = u^{-1}((a,\infty) \cap (-\infty, b)) = u^{-1}((a,\infty)) \cap u^{-1}((-\infty, b)).$$
- But $u(a\mathbf{e}) = a$, so
$$u^{-1}((a,\infty)) = u^{-1}((u(a\mathbf{e}),\infty)) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \succ a\mathbf{e}\},$$
this is open because the strict upper contour set of $a\mathbf{e}$ is open whenever $\succsim$ is complete and continuous (make sure you see why).
- An entirely symmetric argument proves that $u^{-1}((-\infty, b))$ is the strict lower contour set of $b\mathbf{e}$, hence it is also an open set.
- Since $u^{-1}((a,b))$ is the intersection of two open sets, it is open. □

Debreu's Representation Theorem

- Back to the theorem

Theorem (Debreu)

Suppose $X \subseteq \mathbb{R}^n$. The binary relation $\succsim$ on X is complete, transitive, and continuous if and only if there exists a continuous utility representation $u : X \to \mathbb{R}$.

Conclusion

- A preference relation that is complete, transitive, and continuous is entirely described by some continuous utility function that represents it.

- The theorem asserts that one of the utility representations for $\succsim$ must be continuous, not that *all* of the utility representations must be continuous.
- Continuity is a cardinal feature of the utility function, not an ordinal feature, since it is not robust to strictly increasing transformations.

Exercise

Construct a preference relation on $\mathbb{R}$ that is not continuous, but admits a utility representation.

Choice Rules and Utility Functions

The induced choice rule for $\succsim$ is $C_{\succsim}(A) = \{\mathbf{x} \in A : \mathbf{x} \succsim \mathbf{y} \text{ for all } \mathbf{y} \in A\}$

Proposition

If $\succsim$ is a continuous preference relation and $A \subset \mathbb{R}^n$ is nonempty and compact, then $C_{\succsim}(A)$ is nonempty and compact.

Proof.

Suppose $\succsim$ is continuous.

- By Debreu's Theorem, there exists some continuous function u representing the preferences.
- One can show (do it as exercise) that

$$C_{\succsim}(A) = \arg\max_{\mathbf{x} \in A} u(\mathbf{x}).$$

- Nonemptiness then follows from continuity of u and the Extreme Value Theorem.
- Compactness follows from the fact $u^{-1}(\cdot)$ is bounded (it is a subset of the bounded set A), and closed (since the inverse image of a closed set under a continuous function is closed).

□

Structural Properties of Utility Functions

Structural Properties of Utility Functions

- The main idea is to understand the connection between properties of preferences and characteristics of the utility function that represents them.

NOTATION:

- From now on, assume $X = \mathbb{R}^n$.
- Remember the notation for vectors: if $x_i \geq y_i$ for each i, we write $\mathbf{x} \geq \mathbf{y}$.

Monotonicity

- Monotonicity says more is better.

Definitions

- A preference relation $\succsim$ is weakly monotone if $\mathbf{x} \geq \mathbf{y}$ implies $\mathbf{x} \succsim \mathbf{y}$.
- A preference relation $\succsim$ is strictly monotone if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ imply $\mathbf{x} \succ \mathbf{y}$.

- In our notation, $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ imply $x_i > y_i$ for some i.

Question

- What does monotonicity imply for the utility function representing $\succsim$?

Monotonicity: An Example

Example

Suppose $\succsim$ is the preference relation on $\mathbb{R}^2$ defined by

$$\mathbf{x} \succsim \mathbf{y} \quad \text{if and only if} \quad x_1 \geq y_1$$

- $\succsim$ is weakly monotone, because if $\mathbf{x} \geq \mathbf{y}$, then $x_1 \geq y_1$.
- It is not strictly monotone, because

$$(1,1) \geq (1,0) \quad \text{and} \quad (1,1) \neq (1,0)$$

- but

$$\text{not}\,[(1,1) \succ (1,0)]$$

since $(1,0) \succsim (1,1)$.

Strict Monotonicity: An Example

The lexicographic preference on $\mathbb{R}^2$ is strictly monotone

Proof: Suppose $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.
Then there are two possibilities:

$$(a): x_1 > y_1 \quad \text{and} \quad x_2 \geq y_2 \quad \text{or} \quad (b): x_1 \geq y_1 \quad \text{and} \quad x_2 > y_2.$$

- If (a) holds, then
  - $\mathbf{x} \succsim \mathbf{y}$ because $x_1 \geq y_1$, and
  - $\mathrm{not}\,(\mathbf{y} \succsim \mathbf{x})$ because neither $y_1 > x_1$ (excluded by $x_1 > y_1$) nor $y_1 = x_1$ (also excluded by $x_1 > y_1$).
- If (b) holds, then
  - $\mathbf{x} \succsim \mathbf{y}$ because either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$, and
  - $\mathrm{not}\,(\mathbf{y} \succsim \mathbf{x})$ because $\mathrm{not}\,(y_1 > x_1)$ (excluded by $x_1 \geq y_1$) and $\mathrm{not}\,(y_2 \geq x_2)$ (excluded by $x_2 > y_2$).
- In both cases, $\mathbf{x} \succsim \mathbf{y}$ and $\mathrm{not}\,(\mathbf{y} \succsim \mathbf{x})$ thus $\mathbf{x} \succ \mathbf{y}$.

Monotonicity and Utility Functions

Definitions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is

- nondecreasing if $\mathbf{x} \geq \mathbf{y}$ implies $f(\mathbf{x}) \geq f(\mathbf{y})$;
- strictly increasing if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ imply $f(\mathbf{x}) > f(\mathbf{y})$.

- Monotonicity is equivalent to the corresponding utility function being nondecreasing or strictly increasing.

Proposition

If u represents $\succsim$, then:

1. *$\succsim$ is weakly monotone if and only if u is nondecreasing;*
2. *$\succsim$ is strictly monotone if and only if u is strictly increasing.*

Proof.

Question 3a. Problem Set 2. □

Next Class

- Structural properties of utility functions Part II.
- Walrasian demand.